JOURNAL OF APPROXIMATION THEORY **12**, 205–209 (1974)

Korovkin Sets in Locally Convex Function Spaces

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Communicated by G. G. Lorentz

1. INTRODUCTION

In 1953 P.P. Korovkin published his now famous result on the convergence of positive operators on $C[a, b]$ and since that time there has been considerable work done to extend and generalize the results of Korovkin. An exposition of these results with references can be found in [2].

Let E be a Banach space and $\mathscr{L}$ some set of bounded linear operators that map E into itself. A subset F (very often finite) of E is called a *Korovkin set* if for each sequence $\{L_n\}$ of operators in $\mathscr{L}$

$$L_n f \to f, \qquad f \in F \tag{1.1}$$

implies

$$L_n f \to f, \qquad f \in E. \tag{1.2}$$

This definition is due to Y. A. Saskin who investigated these sets for $E = C(X)$, X a compact Hausdorff space and the class of positive operators [3]. Korovkin sets have also been studied in $C(X)$ and L_1 with respect to contraction operators [2, 4, 6].

Hithertofore, Korovkin sets have been studied only in a Banach space, in fact for the most part only in $C(X)$ and L_1. It is the purpose of this note to look at Korovkin sets in a locally convex space which is not normable.

Let X be a locally compact, σ-compact, Hausdorff space, $C(X)$ the space of continuous, real-valued functions defined on X. We shall give $C(X)$ the topology of uniform convergence on compact sets. Since X is σ-compact, $C(X)$ is a Frechet space, which implies it is barreled. The dual space is the set of all regular Radon measures on X with compact support [1, p. 203].

For simplicity we shall restrict the discussion to the above-mentioned space, but it should be pointed out that most of the results will hold for any subspace of $C(X)$ which is barreled and whose dual space contains all the point-evaluation functionals. An example of such a subspace is the space E_c of functions having compact support with the inductive topology with respect to the family of compact sets [5]. E_c is an example of a space which is barreled, but not metrizable.

We shall denote the point evaluation functional at x by ϵ_x and the set of positive continuous linear functionals on $C(X)$ by $\mathscr{M}$. F will be a subset of $C(X)$ and M will be the linear hull of F. We define the subset

$$\partial F = \{x \in X\colon \varphi \in \mathscr{M},\ \varphi|_F = \epsilon_x \text{ implies } \varphi = \epsilon_x\}. \tag{1.3}$$

If X is compact, then a sufficient condition that F be a Korovkin set is that $\partial F = X$. This is also necessary if X is metrizable. We shall show essentially the same result holds in the more general setting.

2. The Basic Theorems

We now prove some general convergence theorems. The first result will not be used later but it is of some interest.

Proposition 2.1. *Suppose ∂F is dense in X and $L \in \mathscr{L}$ such that $L|_F = I|_F$. Then $L = I$.*

Proof. If $x \in \partial F$, then $\epsilon_x \circ L|_F = \epsilon_x|_F$ which implies $\epsilon_x \circ L = \epsilon_x$. Thus for f in $C(X)$, $(Lf)(x) = f(x)$, for all x in ∂F, which implies $Lf = f$.

This tells us in particular that there are no positive projections other than the identity from $C(X)$ onto M. Observe also that the σ-compactness of X was not needed.

Since $C(X)$ is barreled, a sequence $\{\varphi_n\}$ in $\mathscr{M}$ is bounded if and only if $\{\varphi_n(f)\}$ is bounded for each f in $C(X)$.

Theorem 2.2. *Let $\{\varphi_n\}$ be a bounded sequence in $\mathscr{M}$. If for some x in ∂F*

$$\varphi_n(f) \to f(x), \qquad f \in F, \tag{2.1}$$

then

$$\varphi_n(f) \to f(x), \qquad f \in C(X). \tag{2.2}$$

Proof. Let $\{\varphi_{n'}\}$ be an arbitrary subsequence of $\{\varphi_n\}$. Since $C(X)$ is barreled, there exists a further subsequence $\{\varphi_{n''}\}$ and φ in $\mathscr{M}$ such that $\varphi_{n''} \to \varphi$. By (2.1), $\varphi|_F = \epsilon_x|_F$ and since $x \in \partial F$, $\varphi = \epsilon_x$. Thus $\varphi_{n''} \to \epsilon_x$ and this implies (2.2).

Let us note that ∂F is the largest subset of X for which Theorem 1.2 is true. If $x \notin \partial F$, then there exists $\varphi \in \mathscr{M}$ such that $\varphi|_F = \epsilon_x$, but $\varphi \neq \epsilon_x$. If we let $\varphi_n = \varphi$, then (2.1) is true, but (1.2) is not.

THEOREM 2.3. *Assume $\{L_n\}$ is a sequence in $\mathscr{L}$ such that for each f in $C(X)$ and x in X, $\{(L_nf)(x)\}$ is bounded. If*

$$(L_nf)(x) \to f(x), \qquad f \in F, \qquad x \in \partial F, \tag{2.3}$$

then

$$(L_nf)(x) \to f(x), \qquad f \in C(X), \qquad x \in \partial F. \tag{2.4}$$

Proof. Let $\varphi_n = \epsilon_x \circ L_n$, $x \in \partial F$ and apply Theorem 2.2.

In particular if $\partial F = X$, then we have a Korovkin type theorem involving pointwise convergence.

We will use the following terminology: A subset A of X is *sequentially compact* provided each sequence in A has a subsequence that converges to an element of A.

Now let $\{L_n\}$ be a sequence in $\mathscr{L}$ with the following boundedness condition:

For each f in $C(X)$ and each sequentially compact subset A of X, $\{(L_nf)(x)\}$ is bounded, uniformly for x in A. (2.5)

We say that F is a *Korovkin set* provided for each sequence $\{L_n\}$ in $\mathscr{L}$ satisfying (2.5), $L_nf \to f$, $f \in F$ implies $L_nf \to f$, $f \in C(X)$.

THEOREM 2.4. *Assume $\{L_n\}$ is a sequence in $\mathscr{L}$ satisfying* (2.5). *Assume for each f in F that*

$$(L_nf)(x) \to f(x), \tag{2.6}$$

uniformly on compact subsets of ∂F. Then for each f in $C(X)$

$$(L_nf)(x) \to f(x), \tag{2.7}$$

uniformly on sequentially compact subsets of ∂F.

Proof. Assume A is a sequentially compact subset of ∂F, $f_0 \in C(X)$ for which (2.7) is false. Then there exists $\epsilon > 0$, $\{L_{n_k}\}$, $x_k \in A$ such that

$$|(L_{n_k}f_0)(x_k) - f_0(x_k)| \geqslant \epsilon, \qquad k = 1, 2,\ldots. \tag{2.8}$$

By passing to a subsequence if necessary we may assume $x_k \to x \in A$. By (2.6), for f in F, $(L_{n_k}f)(x_k) \to f(x)$, i.e., $\epsilon_{x_k} \circ L_{n_k} \to \epsilon_x$ on F. By Theorem 2.2, $\epsilon_{x_k} \circ L_{n_k} \to \epsilon_x$ on $C(X)$, whence $|(L_{n_k}f_0)(x_k) - f_0(x)| \to 0$ as $k \to \infty$. Then the inequality

$$|(L_{n_k}f_0)(x_k) - f_0(x_k)| \leqslant |(L_{n_k}f_0)(x_k) - f_0(x)| + |f_0(x_k) - f_0(x)|$$

yields a contradiction of (2.8).

COROLLARY 2.5. *Assume* $\partial F = X$. *If*

$$L_n f \to f, \quad f \in F, \tag{2.9}$$

then

$$L_n f \to f, \qquad f \in C(X), \tag{2.10}$$

i.e., F is a Korovkin set.

COROLLARY 2.6. *Let X be the set of real numbers,* $F = \{1, x, x^2\}$. *Then F is a Korovkin set.*

Proof. In this case, $\partial F = X$.

Let us note that ∂F is the largest set for which Theorem 2.4 holds. More precisely we have the following:

THEOREM 2.7. *If X is metrizable and* $x_0 \notin \partial F$, *then there exists a sequence* $\{L_n\}$ *of positive, continuous linear operators satisfying* (2.3) *such that* $L_n f \to f$, $f \in F$, *but for some* $f_0 \in C(X)$, $(L_n f_0)(x_0) \not\to f(x_0)$.

Proof. The construction is the same as that in [2, p. 8] and so it will be omitted.

3. VARIATIONS OF THE BASIC THEOREMS

Let Y be another locally compact, σ-compact space. $C(Y)$ will also have the topology of uniform convergence on compact sets. We let $\mathscr{L}$ be the set of all positive, continuous linear operators mapping $C(X)$ into $C(Y)$. The basic theorems in Section 2 admit a slight generalization with operators L_n belonging to $\mathscr{L}$, and the identity operator replaced by a "reshuffling" operator L of the same class. This means for each $y \in Y$, there exists an $x \in X$ with $(Lf)(y) = f(x)$ for all f in $C(X)$. Let T: $Y \to X$ be defined by $Ty = x$ if and only if $(Lf)(y) = f(x)$ for all f in $C(X)$.

Henceforth we assume $\{L_n\}$ is a sequence in $\mathscr{L}$ satisfying the following boundedness condition:

For each f in $C(X)$, $\{(L_n f)(y)\}$ is uniformly bounded on each sequentially compact subset of Y. (3.1)

The proofs of the next two theorems are similar to the proofs given in Section 2.

THEOREM 3.1. *Assume* $F \subset C(X)$ *and* $y \in T^{-1}(\partial F)$. *If*

$$(L_n f)(y) \to (Lf)(y), \qquad f \in F, \tag{3.2}$$

then

$$(L_n f)(y) \to (Lf)(y), \qquad f \in C(X). \tag{3.3}$$

THEOREM 3.2. *Assume for each f in F*

$$(L_n f)(y) \to (Lf)(y), \tag{3.4}$$

uniformly on compact subsets of $T^{-1}(\partial F)$. *Then for each f in* $C(X)$

$$(L_n f)(y) \to (Lf)(y), \tag{3.5}$$

uniformly on sequentially compact subsets of $T^{-1}(\partial F)$.

As an application of Theorem 3.2 we obtain a generalization of a theorem which was proved in [2] in a different way.

THEOREM 3.3. *Suppose* $\partial F = X$, *so that F is a Korovkin set for* $C(X)$. *If Y is a subset of X, then the restrictions* f' *of f in F to Y form a Korovkin set* F' *in* $C(X')$.

Proof. Let L be the operator of restriction. Then from Theorem 3.2 it follows, if L_n' map $C(X')$ into itself, and if $L_n'f' \to f'$ for all f' in F', that for all f in $C(X)$, $(L_n' \circ L)(f) \to Lf$ and this means that $L_n'f' \to f'$ for all f' in $C(X')$.

ACKNOWLEDGMENT

I am grateful to Professor G. G. Lorentz for many helpful suggestions concerning this paper.

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